Solution to Homework Assignment No. 2

1. (a) Yes, they form a subspace. Consider $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}$ which are symmetry matrices. That is to say $\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{B}^T = \boldsymbol{B}$. We need to check the following two conditions.

- We have
$$(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T = \boldsymbol{A}+\boldsymbol{B}.$$
Therefore, $\boldsymbol{A}+\boldsymbol{B}$ is also a symmetric matrix.
- For any c,
$$(c\boldsymbol{A})^T = c\boldsymbol{A}^T = c\boldsymbol{A}.$$
Therefore, $c\boldsymbol{A}$ is also a symmetric matrix. Since the above two conditions are satisfied, the symmetric matrices in $\boldsymbol{M}$ form a subspace.

(b) Yes, they form a subspace. Consider $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}$ which are skew-symmetry matrices. That is to say $\boldsymbol{A}^T = -\boldsymbol{A}$ and $\boldsymbol{B}^T = -\boldsymbol{B}$. We need to check the following two conditions.

- We have
$$(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T = (-\boldsymbol{A}) + (-\boldsymbol{B}) = -(\boldsymbol{A}+\boldsymbol{B}).$$
Therefore, $\boldsymbol{A}+\boldsymbol{B}$ is also a skew-symmetric matrix.
- For any c,
$$(c\boldsymbol{A})^T = c\boldsymbol{A}^T = -(c\boldsymbol{A}).$$
Therefore, $c\boldsymbol{A}$ is also a skew-symmetric matrix. Since the above two conditions are satisfied, the skew-symmetric matrices in $\boldsymbol{M}$ form a subspace.

(c) No, they do not form a subspace. A counterexample is given as follows. Consider
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$
It is obvious that $\boldsymbol{A}^T \neq \boldsymbol{A}$ and $\boldsymbol{B}^T \neq \boldsymbol{B}$. They are both unsymmetric matrices. However,
$$\boldsymbol{A}+\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
which is a symmetric matrix. Therefore, the unsymmetric matrices in $\boldsymbol{M}$ do not form a subspace.

2. (a) Consider

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \implies x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies (x_1+x_2+x_3)\begin{bmatrix}1\\0\\0\end{bmatrix}+(x_2+x_3)\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies x_1'\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2'\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3'\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

The column space of this matrix is $\mathcal{R}^3$. Therefore, this system has a solution for any (b_1, b_2, b_3).

(b) Consider

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \implies x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies (x_1+x_2+x_3)\begin{bmatrix}1\\0\\0\end{bmatrix}+(x_2+x_3)\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies x_1'\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2'\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3'\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

The column space of this matrix consists of all the linear combinations of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$. Therefore, this system has a solution for any $(b_1, b_2, 0)$. That is, b_3 must be equal to zero.

(c) Consider

$$\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \implies x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies (x_1+x_2+x_3)\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$
$$\implies x_1'\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2'\begin{bmatrix}0\\0\\0\end{bmatrix}+x_3'\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

The column space of this matrix consists of all the linear combinations of $\begin{bmatrix}1\\0\\0\end{bmatrix}$

and $\begin{bmatrix}0\\1\\1\end{bmatrix}$. Therefore, this system has a solution for any (b_1, b_2, b_2). That is, b_3 must be equal to b_2.

3. (a) First, transform $\boldsymbol{A}$ to the reduced row echelon (RRE) form:

$$\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix} \Longrightarrow \begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&3\end{bmatrix} \text{ (subtract } 1\times\text{ row 1)}$$

$$\Longrightarrow \begin{bmatrix}\boxed{\begin{matrix}1\\0\\0\end{matrix}}&\begin{matrix}2\\0\\0\end{matrix}&\boxed{\begin{matrix}0\\1\\0\end{matrix}}&\begin{matrix}0\\2\\0\end{matrix}&\begin{matrix}0\\3\\0\end{matrix}\end{bmatrix} \begin{matrix}\text{(subtract } 2\times\text{ row 2)}\\ \\ \text{(subtract } 1\times\text{ row 2)}\end{matrix}$$

The pivot variables are x_1 and x_3, and the free variables are x_2, x_4, and x_5.

- Given $(x_2, x_4, x_5) = (1, 0, 0)$, we can have $(x_1, x_3) = (-2, 0)$.
- Given $(x_2, x_4, x_5) = (0, 1, 0)$, we can have $(x_1, x_3) = (0, -2)$.
- Given $(x_2, x_4, x_5) = (0, 0, 1)$, we can have $(x_1, x_3) = (0, -3)$.

Therefore, we have

$$\mathcal{N}(\boldsymbol{A}) = \left\{ x_2 \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix} + x_5 \begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix} : \quad x_2, x_4, x_5 \in \mathcal{R} \right\}.$$

(b) First, transform $\boldsymbol{B}$ to the RRE form:

$$\begin{bmatrix}2&4&2\\0&4&4\\0&8&8\end{bmatrix} \Longrightarrow \begin{bmatrix}2&0&-2\\0&4&4\\0&0&0\end{bmatrix} \begin{matrix}\text{(subtract } 1\times\text{ row 2)}\\ \\ \text{(subtract } 2\times\text{ row 1)}\end{matrix}$$

$$\Longrightarrow \begin{bmatrix}\boxed{\begin{matrix}1\\0\\0\end{matrix}}&\boxed{\begin{matrix}0\\1\\0\end{matrix}}&\begin{matrix}-1\\1\\0\end{matrix}\end{bmatrix} \begin{matrix}\text{(divide by 2)}\\ \text{(divide by 4)}\\ \\ \end{matrix}$$

The pivot variables are x_1 and x_2, and the free variable is x_3.

- Given $x_3 = 1$, and we have $(x_1, x_2) = (1, -1)$.

Therefore, we have

$$\mathcal{N}(\boldsymbol{B}) = \left\{ x_3 \begin{bmatrix}1\\-1\\1\end{bmatrix} : \quad x_3 \in \mathcal{R} \right\}.$$

4. By Kirchhoff's Law, we can obtain the following equations:

- For node 1, we have $y_3 = y_1 + y_4$.
- For node 2, we have $y_1 = y_2 + y_5$.

- For node 3, we have $y_2 = y_3 + y_6$.
- For node 4, we have $y_4 + y_5 + y_6 = 0$.

We can rearrange the four equations as

$$\begin{cases} y_1 & & -y_3 & +y_4 & & & = 0 \\ -y_1 & +y_2 & & & +y_5 & & = 0 \\ & -y_2 & +y_3 & & & +y_6 & = 0 \\ & & & -y_4 & -y_5 & -y_6 & = 0 \end{cases}$$

which is equivalent to $\boldsymbol{Ay} = \mathbf{0}$ given by

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then we proceed to find the three special solutions in the nullspace of $\boldsymbol{A}$. First, change $\boldsymbol{A}$ into the RRE form:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \text{(subtract } -1 \times \text{row 1)} \\ \\ \\ \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ \text{(subtract } -1 \times \text{row 2)} \\ \\ \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{(subtract } 1 \times \text{row 3)} \\ \text{(subtract } 1 \times \text{row 3)} \\ \\ \text{(subtract } -1 \times \text{row 3)} \end{matrix}$$

The pivot variables are y_1, y_2, and y_4, and the free variables are y_3, y_5, and y_6.

- Given $(y_3, y_5, y_6) = (1, 0, 0)$, we can have $(y_1, y_2, y_4) = (1, 1, 0)$.
- Given $(y_3, y_5, y_6) = (0, 1, 0)$, we can have $(y_1, y_2, y_4) = (1, 0, -1)$.
- Given $(y_3, y_5, y_6) = (0, 0, 1)$, we can have $(y_1, y_2, y_4) = (1, 1, -1)$.

Finally, the three special solutions in the nullspace of $\boldsymbol{A}$ can be obtained as

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

5. (a) Transform the matrix into the RRE form:

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 6 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \text{(subtract } 1/3 \times \text{row 1)} \\ \text{(subtract } 4/3 \times \text{row 1)} \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(divide by 3)}$$

The original matrix is of rank 1 and can be written as

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}.$$

We now find the nullspace matrix. Since

$$\boldsymbol{R} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

the nullspace matrix is given by

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Transform the matrix into the RRE form:

$$\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 2 & 6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \text{(subtract } -1/2 \times \text{row 1)} \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(divide by 2)}$$

The original matrix is of rank 1 and can be written as

$$\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}.$$

We now find the nullspace matrix. Since

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

the nullspace matrix is given by

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -1 & -3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. (a) Since

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$$

the nullspace matrix is

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}.$$

(b) Since

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

the nullspace matrix is

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}.$$

(c) Since

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$$

the nullspace matrix is

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

7. Consider the augmented matrix and perform elimination, and we have

$$\begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 6 & 2 & 6 \end{bmatrix} \begin{matrix} \\ \text{(subtract } 2 \times \text{row 1)} \\ \text{(subtract } -1 \times \text{row 1)} \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{(subtract } 1 \times \text{row 2)} \\ \\ \text{(subtract } 2 \times \text{row 2)} \end{matrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 3 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{(divide by 3)}$$

The pivot variables are x_1 and x_3, and the free variables are x_2 and x_4. First, we want to find a particular solution. Choose the free variables as $x_2 = x_4 = 0$. Then we have $x_1 = -2$ and $x_3 = 1$. Therefore, a particular solution is

$$\boldsymbol{x}_p = (-2, 0, 1, 0).$$

Then we want to find the nullspace vectors $\boldsymbol{x}_n$.

- Given $(x_2, x_4) = (1, 0)$, we can have $(x_1, x_3) = (-3, 0)$.
- Given $(x_2, x_4) = (0, 1)$, we can have $(x_1, x_3) = (-1, -1/3)$.

Therefore, we can obtain

$$\boldsymbol{x}_n = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

where $x_2, x_4 \in \mathcal{R}$. Finally, the complete solution is given by

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

where $x_2, x_4 \in \mathcal{R}$.

8. (a) Since this system has only one special solution, the nullspace solution is

$$\boldsymbol{x}_n = x_k \boldsymbol{s}$$

where x_k is the free variable. The nullspace solutions $\boldsymbol{x}_n$ form a line, and we can know that $\boldsymbol{A}$ is with full row rank $r = m = 4 - 1 = 3$.

(b) For $\boldsymbol{s} = (2, 3, 1, 0)$, the last zero restricts the variable x_4 to be a pivot variable. Therefore, the RRE form $\boldsymbol{R}$ is

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Since this matrix $\boldsymbol{A}$ is with full row rank $r = m = 3$, we know that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$.